THE

MATHEMATICAL GAZETTE.

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.; E. T. WHITTAKER, M.A.; W. E. HARTLEY, B.A.

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THE DEVELOPMENT OF GEOMETRICAL METHODS.

V.

FROM this moment a brilliant period opens for geometrical research of every kind. Analysts interpret all their results, and set to work to translate them by constructions. Geometers endeavour to discover in every question some general principle—in most cases impossible to prove without the aid of Analysis—so as to deduce a host of particular consequences, closely connected with each other and with the principle from which they are derived. Jacobi's brilliant pupil, Otto Hesse, admirably develops the methods of homogeneous co-ordinates of which Plücker, perhaps, did not fully appreciate the value. Boole discovers in Bobillier's polars the first notion of a covariant; the theory of forms is created by the labours of Cayley, Sylvester, Hermite, and Brioschi. Aronhold, Clebsch, Gordan, and other geometers still with us, invent the definitive notation of the theory, establish the fundamental theorem relative to the limitation of the number of covariant forms, and thus succeed in giving to the theory its fullest extension and scope.

The theory of surfaces of the second order, constructed mainly by the school of Monge, is enriched by a large number of elegant properties—mostly due to Hesse, who later finds in Paul Serret a worthy rival and an investigator who will continue his work.

The properties of the polars of algebraical curves are developed by Plücker, and in particular, by Steiner. The study of curves of the third order, which have by this been the subject of research for a considerable time, is enriched and rejuvenated by a host of new ideas. In the first place, Steiner treats by pure geometry the double tangents of curves of the fourth order, and Hesse, following in his steps, applies algebraical

methods to this beautiful problem, as well as to the points of inflexion of curves of the third order.

The idea of *class*, introduced by Gergonne, the study of a paradox partially elucidated by Poncelet, relating to the respective degrees of two curves each the polar reciprocal of the other, are the immediate impulse to Plücker's researches on the so-called *ordinary* singularities of algebraical plane curves. The celebrated formulæ discovered by Plücker are later extended by Cayley and by other geometers to algebraical gauche curves, and by Cayley again and by Salmon to algebraical surfaces. Singularities of a higher order are in their turn attacked by geometers; Halphen shews that, contrary to an opinion which then widely obtained, each of these singularities may be considered as equivalent to a certain group of ordinary singularities, and his researches bring to a close for the time being this difficult and important question.

Analysis and Geometry—Steiner, Cayley, Salmon, Cremona—meet in their investigations into surfaces of the third order; and as Steiner foresaw, this theory becomes as simple and as easy as that of surfaces of the second order.

Ruled algebraical surfaces, so important in their applications, are studied by Chasles, by Cayley, the marks of whose influence are seen in every form of mathematical investigation, by Cremona, Salmon, and La Gournerie; and later by Plücker in a volume to which I shall presently recur.

The study of the general surface of the fourth order still appears to be too difficult; but that of the particular surfaces of that order with multiple points or multiple lines is begun by Plücker for wave surfaces, by Steiner, Kummer, Cayley, Moutard, Laguerre, Cremona, and many others. As for the theory of algebraical gauche curves, its elements are extended, and finally it receives through the investigations of Halphen and Noether the most notable advancement. Between their labours it is impossible on such an occasion as this to draw a distinction. A new theory with a great future before it is initiated by Chasles, Clebsch, and Cremona; it deals with all the algebraical curves which can be traced on a given surface.

Homography and correlation, the two methods of transformation which originated all the preceding investigations, receive in their turn an unexpected development; they are not the only methods which bring a single element into correspondence with a single element, as is shewn by a particular transformation briefly pointed out by Poncelet in his *Traité des propriétés projectives*. Plücker invents *transformation by reciprocal radii vectores* or *inversion*, and it was not long before Sir William Thomson and Liouville shewed its importance both in Mathematical

Physics and in Geometry. Magnus, a contemporary of Möbius and Plücker, maintained that he had found the most general transformation which will bring a point into correspondence with a point. But Cremona's researches shew that Magnus transformation is only the first of a series of birational transformations which the great Italian geometer shews us how to obtain methodically, at any rate for the figures of Plane Geometry. Cremona's transformations will retain their interest for a long time to come, although later researches proved that they always reduce to a series of successive applications of the transformation we owe to Magnus.

VI.

All the investigations I have mentioned, and others to which I shall presently recur, originate and find their first impulse in the conceptions of modern geometry. But I must now point out another stimulus to great progress in geometrical research. Legendre's Theory of Elliptic Functions, which had been too much neglected by French geometers, was developed and extended by Abel and Jacobi. With these great geometers, almost immediately followed by Weierstrass and Riemann, the theory of Abelian functions, which Algebra will presently attack by its own unaided resources, will bring to the Geometry of curves and surfaces a contribution the importance of which will continue to increase.

Jacobi had already used the analysis of elliptic functions to prove Poncelet's celebrated theorems on in- and circum-scribed polygons, thus opening a chapter which has been enriched by a large number of elegant results; by methods connected with geometry, he had also succeeded in integrating Abelian equations.

But Clebsch was the first to shew in a long series of investigations the whole importance of the notion of the *deficiency* of a curve, due to Abel and Riemann, by developing a vast number of results and elegant solutions, which the use of Abelian integrals appeared, so simple was it, to connect with their real point of departure. The study of the points of inflection of curves of the third order, that of double tangents of curves of the fourth order, and, in general, the theory of osculation on which the ancients and moderns had so often exercised their minds, were connected with the beautiful problem of the division of the elliptic and Abelian functions.

In one of his memoirs, Clebsch had discussed *rational* curves, or curves of deficiency zero; this led him, towards the end of a life alas! too short, to consider what may be called *rational* surfaces, those which can be simply represented by a plane. Here was a vast field for research, already opened for the elementary cases by Chasles, in which Clebsch was followed by

Cremona and many other savants. In this connection Cremona, generalising his researches in plane geometry, brought to light not the whole of the birational transformations of space, but some of the most interesting of these transformations. The extension of the idea of deficiency to algebraical surfaces has already begun; already also work of great value has shewn that the theory of simple or multiple integrals of algebraical differentials will find, in the study of curves and surfaces alike, an extensive field of important applications; but it is not for me to dwell any further on this topic.

VII.

While the mixed methods of which we have just pointed out the principal applications were being created, pure geometers did not remain inactive. Poinsot, the creator of the theory of couples, developed by a purely geometrical method, described by him as "one in which the object of research is never for one moment out of sight," the theory of rotation of a solid body which the investigations of d'Alembert, Euler, and Lagrange seemed to have exhausted; Chasles made a valuable contribution to kinematics by his beautiful theorems on the theory of the displacement of a solid body, which have since been extended by other elegant methods to the case in which the motion has various degrees of freedom. He published his beautiful theorems in the theory of attraction which are worthy to rank with those of Green and Gauss. Chasles and Steiner found themselves on common ground in the study of the attraction of ellipsoids, and thus shewed once more that geometry has its place marked out for it in the most important questions of the integral calculus.

Steiner did not disdain to devote his attention at the same time to the elementary portion of geometry. His researches on the contacts of circles and conics, on isoperimetrical problems, on parallel surfaces, on the centre of gravity of curvature,* aroused the admiration of all by their simplicity and their depth.

Chasles introduced his principle of correspondence between two variable objects, which has given birth to so many applications; but here analysis assumed its rights, and studied the essentials of the principle, giving it precision, and generalising it. This was also the case with the famous theory of *characteristics* and the numerous researches of Jonquières, Chasles, Cremona, and others, who were to lay the foundation of a new branch of geometry—*Enumerative Geometry.* For several years the validity of Chasles' celebrated postulate was unchallenged; many geometers thought that it had been irrefutably established. But, as Zeuthen observed at the time, it is very difficult in proofs of

* *Krummungsschwerpunkt* [Tr.].

this kind to recognise that there is always some weak point which the author may not have perceived; in fact, Halphen, after many fruitless attempts, definitively crowned all these researches by clearly indicating in what cases Chasles' postulate may be accepted, and when it may be rejected.

VIII.

These are the chief investigations which restored synthetic geometry to its place of honour, and assured to it in the course of the last century its due place in mathematical research. Many illustrious workers have taken part in this great geometrical movement, but it must be recognised that their leaders and guides were Chasles and Steiner. Such was the splendour of the light cast by their wonderful discoveries that, at any rate for the moment, they threw into the shade the publications of other modest geometers, who were perhaps less desirous of discovering brilliant applications likely to kindle a love for geometry, than of building the science itself upon an absolutely solid foundation. Perhaps the labours of the latter received a more tardy reward; but their influence increases daily, and no doubt will increase still more. There is no doubt that were I to pass them by in silence, I would be neglecting one of the principal factors which will play their part in future researches. It is especially to von Staudt that I am here alluding. His geometrical work has been expounded in two works of great interest: the *Geometrie der Lage*, which appeared in 1847, and the *Beiträge zur Geometrie der Lage*, published in 1856, that is to say, four years after the *Géométrie Supérieure*.

Chasles, as we have seen, was attempting to build up a body of doctrine independent of the Cartesian analysis, and had not completely succeeded. I have already pointed out one of the reproaches that may be levelled at this system; imaginary elements are only defined in it by their symmetrical functions, which necessarily excludes them from a considerable number of investigations. On the other hand, the constant use of anharmonic ratio, transversals, and involution, requiring frequent analytical transformations, gives to his *Géométrie Supérieure* an almost exclusively metrical character, which notably differentiates it from the methods of Poncelet. Returning to these methods, von Staudt endeavoured to construct a geometry free from all metrical relations, and exclusively based upon relations of situation. It is in this spirit that his first work, the *Geometrie der Lage* (1847) was conceived. The author takes as his point of departure the harmonic properties of the complete quadrilateral and those of homologous triangles, proved solely by considerations of geometry of three dimensions, analogous to those of which the school of Monge has made so frequent a use.

In the first part of his work, von Staudt entirely omitted imaginary relations. It is only in his *Beiträge*, his second work, that, by a very original extension of Chasles's method, he geometrically defined an isolated, imaginary element and distinguished it from its conjugate. This extension, although rigorous, is laborious and very abstract. It may be in substance defined as follows: two conjugate imaginary points may always be considered as the double points of an involution on a real line; so that we pass from an imaginary to its conjugate by changing i into $-i$, and as we may distinguish the two imaginary points by making correspond to each of them one of the two different directions that may be attributed to the line. There is something rather artificial in this; the development of the theory erected on such foundations is necessarily complicated. By purely projective methods, von Staudt established a complete method for calculating the anharmonic ratios of the most general imaginary elements. As in the case of all geometry, projective geometry employs the notion of order, and order involves number. It is not therefore surprising that von Staudt was obliged to build up in detail his method of calculation, but the ingenuity which he displayed in arriving at his conclusions must be admired. In spite of the efforts of the distinguished geometers who have attempted to simplify its exposition, we fear that this part of von Staudt's geometry, like the otherwise so interesting geometry of that profound thinker Grassmann, will not prevail against the analytical methods which have now come into almost universal favour. Life is short; geometers know and also practise the principle of least action. In spite of these fears, which should discourage no one, it seems to me that, in the first form that was given to it by von Staudt, projective geometry must necessarily become the companion of descriptive geometry; that the one is called upon to revive the other in spirit, processes, and applications. This has already been felt in several countries, and notably in Italy, where the great geometer Cremona did not disdain to write for schools an elementary treatise on projective geometry.

IX.

In the preceding sections I have tried to follow and to exhibit clearly the remotest results of the methods of Monge and Poncelet. By inventing tangential and homogeneous coordinates, Plücker had seemed to exhaust all that the method of projections and of polar reciprocals could supply to analysis. Towards the end of his life he returned to his early researches, and gave them an extension which was to widen in unexpected directions the domain of geometry.

Preceded by innumerable researches on systems of straight lines by Poinsot, Möbius, Chasles, Dupin, Malus, Hamilton, Kummer, Transon, and especially by Cayley, who was the first to introduce the notion of line coordinates,—researches which originated either in statics and kinematics, or in geometrical optics,—Plücker's line geometry will always be regarded as the part of his work in which we meet with the newest and most interesting of his ideas. It is important that Plücker should have been the first to build up a methodical treatment of the straight line as an element, but that is nothing compared to what he discovered. It is sometimes said that the principle of duality brings into evidence the fact that the plane as well as the point may be considered as an element of space. That is true, but by adding to the plane and the point the straight line as a possible element of space, Plücker was led to recognise that any curve or surface may also be considered as elements of space, and thus sprang into existence a new Geometry, which has already inspired a large number of treatises, and which will stimulate even more in the future. A beautiful discovery of which I shall speak further on has already connected the geometry of spheres and that of straight lines, and has suggested the introduction of the notion of the coordinates of a sphere. The theory of systems of circles has already commenced. No doubt it will be developed with the detailed investigation of the representation, which we owe to Laguerre, of an imaginary point in space by an orientated circle. But before describing the development of these new ideas which have given fresh life to the infinitesimal methods of Monge, I must retrace my steps for a moment, to touch on the history of the branches of geometry which I have so far neglected.

X.

Among the works due to the school of Monge, I limit myself here to the consideration of those which are connected with *finite* geometry; but some of his pupils devoted themselves in particular to the development of the new ideas of infinitesimal geometry, which were utilised by their master in the study of curves of double curvature, lines of curvature, and the generation of surfaces—ideas which are partially expounded in the *Application de l'Analyse à la Géométrie.* Among them we may mention Lancret, the author of beautiful researches on gauche curves, and especially Charles Dupin, the only one, perhaps, who followed all the paths which were opened up by Monge.

Among other works, we owe to Dupin two volumes, the authorship of which Monge would not have been ashamed to acknowledge: the *Développements de Géométrie pure*, which

appeared in 1853, and the *Applications de Géométrie et de Mécanique* which dated from 1822. In them we find the idea of the *indicatrix* which was, according to Euler and Meunier, to revolutionise the whole theory of curvature, the conceptions of conjugate tangents and of asymptotic lines which have taken so important a place in recent researches. We cannot forget the determination of the surface, the lines of curvature of which are circles; nor, in particular, the Memoir on triply orthogonal surfaces, in which is to be found, together with the discovery of the triple system formed of surfaces of the second degree, the celebrated theorem with which the name of Dupin will always be connected.

Under the influence of these works, and of the revival of synthetic methods, the geometry of the infinitely small resumed in all researches the permanent position which Lagrange had hoped to mark out for it. It is a remarkable thing that the geometrical methods thus revived were to receive their liveliest impulse on the publication of a memoir which at first, at least, seemed to be connected with analysis pure and simple. I mean Gauss's celebrated volume *Disquisitiones generales circa superficies curvas*, which was published in 1827 at Göttingen, and the appearance of which may be said to mark an epoch in the history of infinitesimal geometry.

From this moment the infinitesimal method received such an impulse in France as it had not yet experienced. Frenet, Bertrand, Molins, J. A. Serret, Bouquet, Puiseux, Ossian Bonnet, and Paul Serret developed the theory of gauche curves. Liouville, Chasles and Minding combined with them in methodically following up Gauss's memoir. Jacobi's integration of the differential equation of the geodesic lines of the ellipsoid inspired many investigations. At the same time the problems studied in Monge's *Applications de l'Analyse* were largely developed. The determination of all surfaces having their lines of curvature plane or spherical completed in the most happy manner some of the partial results already obtained by Monge. It was now that one whom Jacobi considered the most penetrating of geometers, Gabriel Lamé (who like Charles Sturm had begun with pure geometry, and had already made the most interesting contributions to that science in a little work published in 1817, and in Memoirs which appeared in the *Annales de Gergonne*), utilised the results obtained by Dupin and Binet on the system of confocal surfaces of the second degree. He, conceiving the idea of curvilinear co-ordinates of space, created an entirely new theory which was destined to receive applications of the most varied kind in mathematical physics.

(To be continued.)

ON 'CIRCULAR MEASURE' AND THE PRODUCT FORMS OF THE SINE AND COSINE.

I. CIRCULAR MEASURE.

WERE it not for Euclid's Propositions iii. 26 (proved by superposition) and vi. 33, the statement that he nowhere defines the circumference of a circle or any curved line would surely be a truism. His definition of a circle is substantially only a locus definition. It may well be argued that neither does he define a straight line (or an angle); but in the case of a straight line he introduces sufficient ideas and axioms to make his subsequent propositions convincing, whereas he gives us nothing to bridge the interval between a straight line and a curved one.

And so, wise after the event, we may say that all the thought spent in old days over 'squaring the circle' had no adequate foundation, and was foredoomed to fail.

As to iii. 26 and vi. 33, they are only further examples of the truth with which we are familiar in the case of angles, that it is sometimes easier to prove that things are equal etc., than to say exactly what they are.

The following is the result of an attempt to reach by the simplest means from established straight line results a satisfactory idea of an 'arc of a circle.' The use of the Trigonometrical Functions does not indicate a lack of simplicity, these, as used, being nothing but convenient words for expressing simple straight line ratios.

The following propositions are established in what follows:

I. $n \sin \frac{A}{n}$ increases and $n \tan \frac{A}{n}$ diminishes as n increases, A representing any angle with no reference to its 'measure.'

II. Hence the combined lengths of 'inscribed chords' and of 'circumscribed tangents,' as they increase in number, approach one another (and therefore some fixed quantity) indefinitely. This quantity may be called the length of the arc or the circular measure of the angle subtended.

III. This quantity is proportional to the angle it subtends at 'the centre.'

I. Let Figs. 1 and 2 (p. 130) represent n angles at a point O, each being x, ($nx <$ a right angle), and A, 1, 2, ... n points at unit distance from O along their arms. In Fig. 1 straight lines are drawn from 1, 2, ... n, parallel to OA, and in Fig. 2, $O1$, $O2$ ... On are produced, to meet in both cases the straight line from A at right angles to OA.

Then by Definition, An' is the *sine* of nx, and An'' is the *tangent* of nx.

$$\therefore \frac{\sin nx}{n} \text{ is the average of } A1', 1'2', \ldots$$

and $$\frac{\tan nx}{n} \text{ is the average of } A1'', 1''2'', \ldots.$$

The lines $A1$, 12, 23 ... are of course all equal.

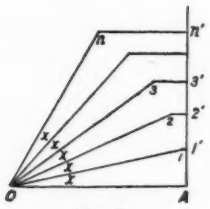

FIG. 1.

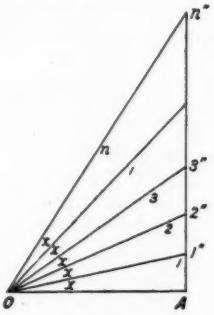

FIG. 2.

Now in Fig. 1 these equal straight lines $A1$, 12 ... make constantly increasing angles with An' and $\therefore$ the intercepts $A1'$, $1'2'$, etc., constantly diminish.

$\therefore$ their average diminishes as n increases.

$$i.e.\ \frac{1}{n}\sin nx \text{ diminishes as } n \text{ increases.}$$

But in Fig. 2, the equal lines being at a constant angle with the rays from O, the lines $A1''$, $1''2''$, $2''3''$ ... constantly increase.

$\therefore$ their average increases as n increases.

$$i.e.\ \frac{1}{n}\tan nx \text{ increases with } n.$$

Writing

$\frac{A}{n'}$ for nx, and therefore $n'\frac{x}{A}$ for $\frac{1}{n'}$, n' diminishes as n increases;

$\therefore$, provided $\frac{A}{n'} <$ a rt. $\angle$,

$$n' \sin \frac{A}{n'} \text{ increases, and } n' \tan \frac{A}{n'} \text{ diminishes, as } n' \text{ increases.}$$

II. Now $$\sin \frac{A}{n} = \tan \frac{A}{n} \cdot \cos \frac{A}{n}, \text{ and } \cos \frac{A}{n} < 1.$$

$$\therefore\ n \sin \frac{A}{n} \text{ is always less than } n \tan \frac{A}{n}.$$

$$\therefore\ n\sin\frac{A}{n} \text{ and } n\tan\frac{A}{n} \text{ approach one another.}$$

Further, their difference is

$$n\tan\frac{A}{n}\left(1-\cos\frac{A}{n}\right)=n\tan\frac{A}{n}\,.\,2\sin^2\frac{A}{n}=2n\tan^3\frac{A}{n}\,.\,\cos^2\frac{A}{n},$$

which $$<2n\tan^3\frac{A}{n} \text{ or } \frac{2}{n^2}\left(n\tan\frac{A}{n}\right)^3.$$

Now when n is large, $n\tan\frac{A}{n}<p\tan\frac{A}{p}$, where p is anything less than n; and for all finite values of A, p can be chosen so that $p\tan\frac{A}{p}$ is finite.

$$\therefore \text{ the difference } <\frac{2}{n^2}\times(\text{a finite quantity}),$$

and $\therefore$ diminishes indefinitely as n increases.

$\therefore$ $n\sin\frac{A}{n}$ and $n\tan\frac{A}{n}$ approach one another continuously, and $\therefore$ both approach some fixed quantity a, which is *defined* by this continuous approach of $n\sin\frac{A}{n}$ and $n\tan\frac{A}{n}$.

But in Fig. 1 (with all restrictions as to the size of the angles removed, except that $\frac{A}{2n}<$ a rt. $\angle$) the sum of $A1$, $12\ldots$ is $n\,.\,2\sin\frac{A}{2n}$ if the whole angle is called A, and, if lines are drawn at A, 1, 2 ... at rt. $\angle^s$ to OA, $O1$, $O2\ldots$ to meet, the sum of these is $2n\tan\frac{A}{2n}$.

$\therefore$ a lies between the sums of these 'inscribed chords' and 'circumscribed tangents.'

The limit of either of these sums of straight lines is what we mean by the 'arc of the unit circle' from A to n, or the 'circular measure' of the angle A.

III. To prove that the quantity a (*i.e.* the length of the 'arc') is proportional to A.

Since $$a<\left|\begin{matrix} n\tan\frac{A}{n} \\ n\sin\frac{A}{n}\end{matrix}\right| \text{ and } \beta<\left|\begin{matrix} n'\tan\frac{B}{n'} \\ n'\sin\frac{B}{n'}\end{matrix}\right|,$$

however great n and n' may be,

$$\frac{\alpha}{\beta} < \frac{n}{n'} \frac{\sin\frac{A}{n}}{\sin\frac{B}{n'}} \left| \begin{matrix} \sec\frac{A}{n} \\ \cos\frac{B}{n'} \end{matrix} \right|.$$

Now assume n and n' to be so chosen that

$$\frac{A}{n} = \frac{B}{n'}; \text{ then } \frac{n}{n'} = \frac{A}{B}$$

(the actual magnitude of n and n' remaining free);

$$\therefore \frac{\alpha}{\beta} < \frac{A}{B} \left| \begin{matrix} \sec\frac{A}{n} \\ \cos\frac{A}{n} \end{matrix} \right|, \text{ however great } n \text{ may be.}$$

Now the difference

$$\sec\frac{A}{n} - \cos\frac{A}{n} = \frac{\sin^2\frac{A}{n}}{\cos\frac{A}{n}} = \tan^2\frac{A}{n} \cdot \cos\frac{A}{n} = \frac{1}{n^2}\left(n\tan\frac{A}{n}\right)^2 \cos\frac{A}{n}$$

which is seen, as above, to diminish indefinitely as n increases.

And $\therefore$ the only quantity which can be said to lie between

$$\frac{A}{B} \left| \begin{matrix} \sec\frac{A}{n} \\ \cos\frac{A}{n} \end{matrix} \right|,$$

when n is indefinitely increased, is $\frac{A}{B}$.

$$\therefore \frac{\alpha}{\beta} = \frac{A}{B} \text{ (for all values of } A \text{ and } B).$$

This proves that α as defined in § II. above is a sound 'measure' of the angle A.

[NOTE 1.—Any difficulty that may seem to arise from incommensurable angles is covered by the *meaning* of $\frac{A}{B}$ in such a case. We take it to mean 'something which lies between $\frac{A'}{B}$ and $\frac{A''}{B}$, where A', A'' are angles which are commensurable with B,' one being greater and the other less than A, but both indefinitely near to A, $\frac{\alpha}{\beta}$ having a similar meaning, we cannot escape the

conclusion that with these definitions $\frac{\alpha}{\beta}=\frac{A}{B}$, as for commensurables.]

[NOTE 2.—The 'circular measure' of an angle is perhaps better named 'the arc' of the angle. All measurements of an angle are circular, the length of the arc being compared in the rectangular systems with the length of the circumference, and in 'circular measure' with the length of the radius. The use of 'number of radians' as a definition seems unnatural and unsatisfactory. The *radian* is an uninteresting angle—Lord Kelvin introduced the word merely as a convenience in lecturing, to avoid the long phrase 'angle whose circular measure is.']

II. THE PRODUCT FORMS OF THE SINE AND COSINE.

If n is a positive integer, the addition formulae give $\cos nx$ as an algebraic function of degree n in $\cos x$ only.

Hence

$$\cos 2nx-\cos 2n\alpha \equiv \text{C.}^*(\cos x-\cos x_1)(\cos x-\cos x_2)\ldots(\cos x-\cos x_n),$$

where $\cos x_1$, $\cos x_2$, ..., represent the roots (not yet proved real) of

$$\cos 2nx=\cos 2n\alpha.$$

Now this equation is satisfied for all values of x for which $2nx=4i+2n\alpha$, the angles being measured in right angles and i representing any integer. Thus $x=\frac{2i}{n}+\alpha$.

But the $2n$ values of x given by $i=0, 1, 2, \ldots, 2n-1$ have all different cosines, as the sum of any two angles is $\frac{2i+2j}{n}+2\alpha$, and the difference is $\frac{2i-2j}{n}$ and none of these is a multiple of 4, since $i-j<2n$;

$\therefore$ the $2n$ values of $\cos\left(\frac{2i}{n}+\alpha\right)$ are all roots of $\cos 2nx=\cos 2n\alpha$,

and $\therefore$ $\cos 2nx-\cos 2n\alpha \equiv \text{C.}\prod_{0}^{2n-1}\left[\cos x-\cos\left(\frac{2i}{n}+\alpha\right)\right]$.

Now to obtain $\sin nx$, put $\alpha=0$ in this identity.

Then $\cos 2nx-1$, and therefore

$$\sin^2 nx \equiv \text{C.}\prod_{0}^{2n-1}\left[\cos x-\cos\frac{2i}{n}\right].$$

Now $\cos\frac{2i}{n}=\cos\frac{2j}{n}$ if $2i+2j=4n$, *i.e.* $j=2n-i$;

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* C. is used throughout merely for the word 'constant': its value changes from line to line.

$\therefore$ the pairs of values of i, $\underset{2n-1'}{1}$ $\underset{2n-2'}{2}$ $\cdots$, $\underset{n+1'}{n-1}$ give equal cosines. $i=0$ and $i=n$ are peculiar terms, and

$$\sin^2 nx \equiv \text{C.}\ (\cos x-1)(\cos x+1).\prod_{1}^{n-1}\left(\cos x-\cos\frac{2i}{n}\right)^2;$$

$$\therefore\ \sin nx \equiv \text{C.}\ \sin x.\prod_{1}^{n-1}\left(\cos x-\cos\frac{2i}{n}\right).$$

Again $\cos\dfrac{2i}{n}=-\cos\dfrac{2j}{n}$ if $2i+2j=2n$; $\therefore\ j=n-i$;

$\therefore$ the pairs of values of i, $\underset{n-1'}{1}$ $\underset{n-2'}{2}$ $\cdots$, give opposite cosines, and $i=\dfrac{n}{2}$ is a peculiar term if n is even;

$\therefore\ \sin nx \equiv \text{C.}\ \sin x.\prod_{2}^{n-1\text{ or }2}\left(\cos^2 x-\cos^2\dfrac{\text{even}}{n}\right)$, with an extra factor $\cos x$ if n is even.

But $\cos^2 x-\cos^2\dfrac{\text{even}}{n}=\sin^2\dfrac{\text{even}}{n}-\sin^2 x$, and also can be put in the form

$$\cos^2 x\sin^2\frac{\text{even}}{n}-\sin^2 x.\cos^2\frac{\text{even}}{n}$$

or

$$\cos^2 x.\cos^2\frac{\text{even}}{n}\left(\tan^2\frac{\text{even}}{n}-\tan^2 x\right);$$

$$\therefore\ \sin nx \equiv \text{C.}\ \sin x.\prod_{2}^{n-1\text{ or }2}\left(1-\frac{\sin^2 x}{\sin^2\dfrac{\text{even}}{n}}\right),\ \text{with } \cos x \text{ if } n \text{ is even},$$

and also

$$\equiv \text{C.}\ \tan x.\prod_{2}^{n-1\text{ or }2}\left(1-\frac{\tan^2 x}{\tan^2\dfrac{\text{even}}{n}}\right).\cos^n x.$$

By making x very small we get $\text{C.}=\dfrac{\sin nx}{\sin x}$ or $\dfrac{\tan nx}{\tan x}$, both of which are n.

Now write $\dfrac{x}{n}$ for x.

Then

$$\sin x \equiv n\sin\frac{x}{n}.\prod_{2}^{n-1\text{ or }2}\left(1-\frac{\sin^2\dfrac{x}{n}}{\sin^2\dfrac{\text{even}}{n}}\right),\ \text{with } \cos\frac{x}{n} \text{ if } n \text{ is even},$$

and

$$\equiv n\tan\frac{x}{n}.\prod\left(1-\frac{\tan^2\dfrac{x}{n}}{\tan^2\dfrac{\text{even}}{n}}\right).$$

Now, if ξ is the circular measure of x, that is $\frac{x\pi}{2}$, $\sin\frac{x}{n}<\frac{\xi}{n}$ and $\tan\frac{x}{n}>\frac{\xi}{n}$.

Moreover, if $x<2$, by § II. above,

$$\frac{\sin\frac{x}{n}}{\sin\frac{\text{even}}{n}}>\frac{x}{\text{even}} \quad \text{and} \quad \frac{\tan\frac{x}{n}}{\tan\frac{\text{even}}{n}}<\frac{x}{\text{even}};$$

$$\therefore \sin x<\xi\,.\prod_{2}^{n-1 \text{ or } 2}\left(1-\frac{x^2}{\text{even}^2}\right)$$

and

$$>\xi\,.\prod_{2}^{n-1 \text{ or } 2}\left(1-\frac{x^2}{\text{even}^2}\right)\cdot\left(\cos\frac{x}{n}\right)^n;$$

$$\therefore \sin x<\xi\left(1-\frac{x^2}{2^2}\right)\left(1-\frac{x^2}{4^2}\right)\ldots\left(1-\frac{x^2}{\text{even}^2}\right)\left|\frac{1}{\left(\cos\frac{x}{n}\right)^n}\right|,$$

where n is either of the two numbers succeeding the last 'even' of the product.

Now $\left(\cos\frac{x}{n}\right)^n>\left(1-\frac{\xi^2}{2n^2}\right)^n$, which $>1-\frac{\xi^2}{2n}$;

$\therefore$ finally, $\sin x=\frac{\pi}{2}\,.\,x\left(1-\frac{x^2}{2^2}\right)\left(1-\frac{x^2}{4^2}\right)\ldots\left(1-\frac{x^2}{\text{even}^2}\right)\cdot\left[1-\theta\frac{\xi^2}{2n}\right]$,

where θ is a proper fraction;

$\therefore$ the greater n is, the nearer will $\sin x$ be to the product, and the error gets smaller and smaller without limit.

We now proceed to show that the limitation $x<2$ above can be removed.

If $x>2$, the factors

$$\left(1-\frac{\sin^2\frac{x}{n}}{\sin^2\frac{2}{n}}\right), \quad \left(1-\frac{\tan^2\frac{x}{n}}{\tan^2\frac{2}{n}}\right)$$

are not respectively less and greater than $\left(1-\frac{x^2}{2^2}\right)$; but (assuming n so great that $\frac{x}{n}<1$) both these factors are, for $x>2$, negative, and since now, by § II. above,

$$\frac{\sin\frac{x}{n}}{\sin\frac{2}{n}}<\frac{x}{2} \quad \text{and} \quad \frac{\tan\frac{x}{n}}{\tan\frac{2}{n}}>\frac{x}{2},$$

the corresponding positive factors

$$\left(\frac{\sin^2\frac{x}{n}}{\sin^2\frac{2}{n}}-1\right),\quad \left(\frac{\tan^2\frac{x}{n}}{\tan^2\frac{2}{n}}-1\right)$$

are less and greater than $\left(\frac{x^2}{2^2}-1\right)$, and so for $x>4,\ 6,\ \ldots.$

The final result is therefore true for all values of x.

For the cosine,

put $na=1$ in (I.), then $\cos 2na=\cos 2=-1$;

$$\therefore\ \cos 2nx+1\equiv \text{C.}\prod_{0}^{2n-1}\left[\cos x-\cos\frac{2i+1}{n}\right].$$

Now $\cos\frac{2i+1}{n}=\cos\frac{2j+1}{n}$ if $2i+2j+2=4n$; $\therefore\ j=2n-i-1$,

so that the pairs $\begin{matrix}0 & 1\\ 2n-1' & 2n-2'\end{matrix}\ \cdots\ \begin{matrix}n-1\\ n\end{matrix}$, give equal cosines.

$$\therefore\ 2\cos^2 nx\equiv \text{C.}\prod_{0}^{n-1}\left[\cos x-\cos\frac{2i+1}{n}\right]^2;$$

$$\therefore\ \cos nx\equiv \text{C.}\prod_{0}^{n-1}\left(\cos x-\cos\frac{2i+1}{n}\right).$$

Now $\cos\frac{2i+1}{n}=-\cos\frac{2j+1}{n}$ if $2i+2j+2=2n$, *i.e.* $j=n-1-i$,

so that the pairs $\begin{matrix}0 & 1\\ n-1' & n-2'\end{matrix}\ \cdots,$

give opposite cosines, and $i=\frac{n-1}{2}$ is peculiar if n is odd;

$$\therefore\ \cos nx\equiv \text{C.}\prod_{1}^{n-1\text{ or }2}\left(\cos^2 x-\cos^2\frac{\text{odd}}{n}\right),\ \text{with } \cos x \text{ if } n \text{ is odd},$$

$$=\text{C.}\ \Pi\left(1-\frac{\sin^2 x}{\sin^2\frac{\text{odd}}{n}}\right)=\text{C.}\ \Pi\left(1-\frac{\tan^2 x}{\tan^2\frac{\text{odd}}{n}}\right).\cos^n x\ \text{(as before)},$$

with an extra factor $\cos x$ in the former if n is odd.

$$\therefore\ \cos x=\text{C.}\prod_{1}^{n-1\text{ or }2}\left(1-\frac{\sin^2\frac{x}{n}}{\sin^2\frac{\text{odd}}{n}}\right)=\text{C.}\prod_{1}^{n-1\text{ or }2}\left(1-\frac{\tan^2\frac{x}{n}}{\tan^2\frac{\text{odd}}{n}}\right)\cos^n\frac{x}{n}.$$

Putting $x=0$, we see $C=1$;

$$\therefore \text{ as before, } \cos x < \Pi\left(1-\frac{x^2}{\text{odd}^2}\right)\left|\begin{matrix}1\\ \cos^n\frac{x}{n}\end{matrix}\right|,$$

where n is either of the two numbers succeeding the last 'odd' in the product;

$$\therefore \text{ as before, } \cos x=\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{3^2}\right)\ldots\left(1-\frac{x^2}{\text{odd}^2}\right)\cdot\left[1-\theta\frac{\xi^2}{2n}\right],$$

x being rectangular measure, and ξ its circular measure.

Corollary. By comparing these product forms with the expansions, we have

$$(1)\quad \xi\,.\,\Pi\left(1-\frac{x^2}{\text{even}^2}\right)\cdot\left[1-\theta\frac{\xi^2}{2n}\right]\equiv\Sigma\left(\pm\frac{\xi^{\text{odd}}}{\text{odd}}\right)+\frac{\xi^n}{\lfloor n}\left|\begin{matrix}\sin x\\ 0\end{matrix}\right|;$$

$$(2)\qquad \Pi\left(1-\frac{x^2}{\text{odd}^2}\right)\cdot\left[1-\theta\frac{\xi^2}{2n}\right]\equiv\Sigma\left(\pm\frac{\xi^{\text{even}}}{\text{even}}\right)+\frac{\xi^n}{\lfloor n}\left|\begin{matrix}\sin x\\ 0\end{matrix}\right|.$$

From which follow without difficulty

$$\sum_{2}^{n-1\text{ or }2}\frac{1}{\text{even}^2}+\frac{\theta}{2n}\cdot\frac{\pi^2}{4}=\frac{\pi^2}{4\lfloor 3};$$

$$\therefore\quad \sum_{2}^{n-1\text{ or }2}\frac{1}{\text{even}^2}=\frac{\pi^2}{24}\left|\begin{matrix}1\\ 1-\frac{1}{12n}\end{matrix}\right|$$

and

$$\sum_{1}^{n-1\text{ or }2}\frac{1}{\text{odd}^2}=\frac{\pi^2}{8}\left|\begin{matrix}1\\ 1-\frac{1}{n}\end{matrix}\right|.$$

W. N. ROSEVEARE.

MATHEMATICAL NOTE.

162. [X. 5.] *Extract from an old pamphlet on the slide rule.*

The authors introduced a modification of the old Gunter's rule, and remark in their preface, "And what further recommends it to the world is the *Commodiousness* thereof and the *Manner of Use*; for in *other* instruments you are obliged to state the question right to find the answer, but in this if you state the question *improperly* it will come *right* notwithstanding which renders it *advantageous* and *easy* to those that are unskilled in *Arithmetic* or *common slides*."

The pamphlet is intituled: The Art of | Measuring made Easy | By the Help of | a new Sliding-Rule | which | performs the same at one operation | as requires two three or more | on the sliding rules heretofore used | In a Plain Easy and Concise Method | entirely New | By W. Bradford R. Darby and J. Hulls Philomaths all of Campden in Gloucestershire. Worcester. Printed by H. Berrow, in Goose Lane 1754.

Is there not an opening at the present day for an instrument which will furnish the right answer to the wrong question? C. S. J.

REVIEWS.

Quadratic Partitions. By Lieut.-Col. ALLAN CUNNINGHAM, R.E. Pp. xxiii and 266. 12s. 1904. (Francis Hodgson.)

This book mainly consists of tables of the solutions of

$$t^2 + Du^2 = p,$$

where D is any positive or negative number less than 20, and p is a prime number; t and u are the "unknowns" which are given in the tables. For given values of D, and p, the equation may be impossible, and in that case the tables state the fact. If the equation is possible, and D is positive, the Theory of Numbers shews that the solution is unique. But if D is negative and the equation is possible, there is an infinite number of solutions, all of which can be derived from any one of them. In this case the author gives the least solution. The principal table gives the solutions for the important cases $D=1$, 2, and 3, for all values of p up to 100,000. The only previous tables for these values of D are now difficult to obtain, and are of much smaller extent than the present table, being complete only up to about 13,000. The introduction gives an interesting account of some of the uses in the Theory of Numbers to which the tables can be put.

No one can adequately conceive the labour involved in the calculation of these tables, unless he has himself tried to solve such equations. For some values of D, Gauss, Jacobi, and others have found explicit solutions.

For instance, if $p=t^2+u^2$, where t is odd, and $p=4n+1$, then $t \equiv \frac{1}{2}\frac{2n!}{n!\,n!}$, mod. p; this is theoretically important, but (like most direct methods in the Theory of Numbers) quite impracticable for the purpose of calculating t. Consequently the solutions for each p have to be found individually by a process of trial. We would suggest to the author the desirability of putting his methods on record in any future edition or extension of these tables. We are glad to see it stated in the Introduction that the author (who in 1900 published a "Binary Canon," shewing residues of powers of 2, mod. p) has in preparation other tables relating to the Theory of Numbers. A. E. WESTERN.

Leçons de Trigonométrie rectiligne. By C. BOURLET. Pp. 322. 1904. 6 francs. (Paris, Colin & Cie.)

This work is one of the series of elementary mathematical text-books edited by M. Darboux. Its three parts include the properties of triangles and quadrilaterals with their associated circles; there is also an appendix on the graphical representation of complex quantities, De. Moivre's theorem, the trigonometrical functions of submultiple angles, and the solution of binomial, quadratic, and cubic equations.

The book may with advantage be put into the hands of a clever schoolboy who has acquired a considerable knowledge of the subject from more elementary text-books. It is clear, suggestive, and very complete; for instance, three different ways of solving $a\cos x + b\sin x = c$ are discussed at length. Examples for the student are scarce; there are only 105, including those in the appendix. The author begins with definitions of vectors and projections, and develops his subject on the lines suggested in the *Mathematical Gazette* of October, 1904, pp. 82 to 85. Particularly interesting are the chapters on the solution of

trigonometrical equations, the conversion of a sum into a form suitable for logarithmic calculation, and the solution of triangles. The printing and engraving are beyond praise. H. H.

A New Trigonometry for Schools. By W. G. BORCHARDT and A. D. PERROTT. Pp. 400+xii. 1904. 4s. 6d. (G. Bell & Sons.)

This is a rather more conservative treatise than the majority of recent trigonometries, but it contains numerous concessions to modern theories of teaching. The new and old do not appear as yet to agree well together; but nevertheless the book is distinctly good and should be especially useful to boys who intend to make a serious study of mathematics. The explanations are very full and the examples numerous; few pupils will want to read the whole of the book. The expansions for $\sin\theta$ and $\cos\theta$, and a very clear description of the slide rule, bring the treatise to a close. H. H.

Carl Gustav Jacob Jacobi. By LEO KOENIGSBERGER. Pp. xviii.+554. 1904. (Teubner.)

The hundredth anniversary of Jacobi's birth has been celebrated by the appearance of a biography closely following the lines of the author's *Hermann von Helmholtz*, and amply maintaining the high standard of his former work. Throughout the book it is Jacobi as a mathematician who interests the author rather than Jacobi as a man. A short account of each of Jacobi's numerous contributions to science is given in chronological order, and a very welcome summary of his whole work closes a volume which is as attractive as author and publisher can make it.

Jacobi was the son of Jewish parents and, though known at school as an unusually clever boy, did not reveal exceptional mathematical ability till towards the close of his undergraduate days at the University of Berlin. His inaugural dissertation at the end of his University course first brought him prominently forward as a mathematician, and secured for him in 1826 the post of Privatdozent at Königsberg. It was about this time that he formed the friendships with Bessel and others which so profoundly affected his future labours. From the day he went to Königsberg to the day of his death he worked unceasingly at his contributions to Mathematical Science, and his name rapidly became famous throughout Europe. He was promoted in turn to the posts of Professor Extraordinarius and Professor Ordinarius at Königsberg, and in 1844 was made Fellow of the Berlin Academy, that he might recruit his failing health and devote himself still more exclusively to research. He died of smallpox in 1851 after a brief illness.

Jacobi's labours covered a wide range of subjects. In addition to his work in the theory of transcendental functions, differential equations, calculus of variations, and analytical mechanics, with which his name will always be most closely associated, he made valuable contributions to Astronomy, the theory of series, etc. In everything he endeavoured, not so much to solve isolated problems, however attractive, as to find the intimate connexion underlying the whole of mathematical truth. H. H.

A Handy Book of Logarithms, with Practical Geometry Appendix. Pp. 128. 2s. 1904. (Blackie.)

This book consists of tables of logarithms of numbers from 1 to 10,000; hyperbolic logarithms of numbers from 1·01 to 30; circumferences, diameters, and areas of circles; square roots and cubes; lengths of circular arcs, given their degrees or given their heights; areas of circular segments up to a semi-circle; trigonometrical ratios from 0° to 90°, advancing by the sixth of a degree; logarithmic sines, cosines, etc.; the points of the compass, and the reciprocals of numbers from 1 to 1000. The first two tables are to four places of decimals; of the rest some are to three places, some to five, and some to six. The appendix contains some of the simple problems in practical geometry, rules on mensuration, etc. The tables are very clearly printed.

Plane Geometry, Practical and Theoretical. By J. S. MACKAY. Books I., II., and III. **Congruent and Equivalent Figures**, corresponding to Euc. I.-IV. Pp. viii, 237. 2s. 6d. (Chambers.)

A short introduction familiarises beginners with the use of the protractor, etc. Dr. Mackay dryly remarks in his preface that before going among

foreigners it is well to know something of the language the foreigner speaks. Hence the old plan of placing the definitions first is followed by the author. Dr. Mackay has skilfully adapted the corresponding portion of his edition of the Elements to the requirements of modern programmes of geometrical teaching. He has utilised the excellent collection of exercises with which that edition was adorned, and we also find the historical notes which were so interesting and useful a feature. In addition, this little volume is distinguished from all its competitors by the fact that it is the first to contain a sketch of Geometrography. Dr. Mackay is a careful teacher, and will be found a safe guide for the young student. Beginners will not be much disturbed by the faultiness of the logic in the treatment of tangents to circles by the method of limits. We note that Book I. (128 pp.) is published separately at a shilling.

Encyclopédie des Sciences Mathématiques Pures et Appliquées. French edition, published under the direction of Jules Molk. Tome I., Vol. I. Fascicule I. **Arithmétique.**

We heartily welcome the appearance of the French edition of this monumental Encyclopaedia. Not only does it place within the reach of those who are not accomplished German scholars a most valuable storehouse of information of a kind not elsewhere attainable in so convenient and precise a form, but it is in itself evidence of that *entente cordiale* which reigns throughout the scientific world. For the authors of the original articles in the German edition have collaborated with their French *confrères*, have intimated where, on wider consideration, or for the purposes of a French audience, modifications were advisable or necessary; while, on the other hand, the Frenchmen, on their part, have carefully discussed the proposed changes, and, in particular, have added considerably to the bibliographical information, as may be seen by a glance at the footnotes in asterisks which with the bottom of every pages is sown.

The peculiar value of such a work as this at the present time is obvious. Into every department of science the specialist has penetrated; but the departments of science are not cut and dried divisions. No specialist can afford to be ignorant of what is going on even in fields of research that are apparently as wide as the poles asunder as far as his own investigations are concerned. Again and again have the difficulties in the path of an investigator been removed by an appeal to the researches of others in apparently unconnected regions of research. Who would suppose in the early days of Legendre that the theory of elliptic functions would ever play a part in the development of geometry? And do we not see at the present moment how the theory of groups is making its influence felt in the most remarkable manner in almost every department of mathematics? It is clear that the world of mathematical thought must lie under a deep debt of obligation to Prof. Klein and his band of collaborators, who have made it possible for either the normal mathematician or the specialist in a particular field to refer to what has been done by other workers in the developments of the theories in which they have been interested.

In the fascicule which lies before us we have three sections. First we have 63 pages on the fundamental principles of Arithmetic, in which the article originally due to H. Schubert is expounded by Messrs. Tannery and Molk. The idea of the natural number, from Leibnitz to Frege, Peano, and Russell; numeration; direct and inverse operations; operations of every kind, from the simplest, such as additions to Mannoury's hypermultiplication—all in turn are concisely dealt with, and furnished with copious lists for further reference.

The second section treats of the Combinatory Analysis and the Theory of Determinants—Netto's article, expounded by H. Vogt. The final section is not concluded in this part. It deals with irrational numbers and the idea of the limit, and is interpreted by Prof. Molk for the French reader from the German of A. Pringsheim. Here it is sad to notice that although there are plentiful references to English writers in the second section, from Wallis to MacMahon, yet in section iii. the only English name revealed by a cursory glance is that of Newton!

It is hardly necessary to add that we feel assured that as soon as the quality of the work in this encyclopaedia is known to the younger generation of

mathematicians, they will not be happy till they get it. Klein and his colleagues have indeed conferred upon us a boon which will be generously recognised.

Technical Mechanics. By E. R. Maurer. Pp. xvii, 403. $4. 2nd edition, revised and enlarged, 1904. (New York: Wiley & Sons; London: Chapman, Hall.)

We are glad to see that the merits of Professor Maurer's treatise on Technical Mechanics have so rapidly been recognised. We have already commented in favourable terms on the excellent planning of the broad outlines of this book, and on the skill and judgment with which the illustrative examples of practical interest have been selected and arranged (vol. iii., p. 16). The additions to the present edition consist of an appendix on the principle of virtual work as applied in the case of a particle, a system of particles, and to statically indeterminate problems; an appendix containing sections on truss loads, weight of roofing, weight of trusses, weight of snow, wind pressure, computation of apex loads, maximum stresses, classification of frames, and additional remarks on funicular polygons.

Solutions of the Problems and Theorems in Charles Smith's Geometrical Conics. By Charles Smith. Pp. 143. 6s. 1904. (Macmillan.)

Key to Godfrey and Siddons' Geometry. By E. A. Price. Pp. 172. 5s. 1904. (Cambridge University Press.)

Solutions of the Examples in Hall's Graphical Algebra. By H. S. Hall and H. C. Beaven. Pp. 59. 3s. 6d. 1904. (Macmillan.)

Key to Elementary Geometry. By Cecil Hawkins. Pp. 195. 5s. 1904. (Blackie.)

There will be many teachers to whom Mr. Smith's key will be a boon, and many private students who will find it of the greatest value if they have no other guidance. We are glad to see that the price is prohibitive to the ordinary school-boy, although when a boy is old enough to tackle the stiffer examples in Mr. Smith's *Conics* he is quite old enough to know how to use a key wisely and well. We have tested a good few of the solutions and found them neat and elegant, as indeed one would expect from a teacher of Mr. Smith's experience. We may add that the solutions refer to the fourth edition of the *Geometrical Conics*.

We may compliment Mr. Price on the figures in his *Key to Godfrey and Siddons' Geometry*, and, in fact, we ought to extend the compliment to the Cambridge University Press for their share in its production. There must be a very large number of teachers to whom this little book will be of value, partly because practical geometry was not in their mathematical course while at the University and partly because the figures and solutions will save them much time in examining the works of their classes. The figures are reduced copies of diagrams drawn to scale, and in most cases Mr. Price has given the answers to one place of decimals more than is asked for. Here again the price is probably beyond the reach of all beginners but those drawn from the ranks of the plutocracy.

As one might expect, for the present there will be many to whom graphical work is both novel and difficult. Mr. Hall tells us that he has had many letters of enquiry on the subject of "graphics" from both teachers and students. This has naturally led him to prepare a Key with full solutions and diagrams of all the more important examples. If, as he suggests, the Key is only used as a final check on the result, the appearance of this little volume is amply justified. The 75 graphs are very carefully and accurately drawn. We know from personal experience the time and attention that is required in the correction of the beginners' graphs, and we can cordially recommend this Key to all teachers of large classes using the *Graphical Algebra*. Mr. Hawkins gives no diagrams and, as he says, has found it necessary to give fuller treatment to particular cases than is advisable or necessary in ordinary cases. This will not make his book any the less useful to the teacher, for whom the solutions are intended. Ample hints are given for the solutions of many of the questions, but we rather think that the enterprising school-boy of tender years who purchases this Key for the purpose of diminishing his mental exertions will report

to his envious neighbours that he has been "jolly well sold." On the other hand, the solutions are ample for the young teacher or for the more advanced private student who requires some external assistance.

A Short Introduction to Graphical Algebra. By H. S. HALL. Pp. 49. 1s. 1903. (Macmillan.) Second Edition, revised and enlarged.

The Oxford and Cambridge Graphical Algebra. By R. H. ALLPRESS and F. MARSHALL. Pp. 26. 6d. n.d. (Gill.)

An Elementary Treatise on Graphs. By Professor G. A. GIBSON. Pp. x, 183. 3s. 6d. 1904. (Macmillan.)

Graphs and Imaginaries. By J. G. HAMILTON and F. KETTLE. Pp. 41. 1904. (Arnold.)

The *Short Introduction to Graphical Algebra*, already a new issue, is a considerable improvement on the first edition, mainly owing to the inclusion of applications to practical questions preparing the student for his future work in the laboratory and the workshop. It is now an adequate introduction.

Messrs. Allpress and Marshall's little pamphlet is confined to the consideration of the graphs of simple functions, easy maxima and minima, limiting values, and the use of co-ordinates in finding the areas of triangles. It is well done within these narrow limits, but contains no practical examples.

In the case of Professor Gibson's little treatise we come to another type of book. We wish we had space to quote his wise remarks on the place that graphical methods are taking in the mathematical teaching of the day. The general treatment of the subject is excellent, and a large number of the useful examples are taken from laboratory note-books. Cubic and biquadratic, logarithmic, exponential, and trigonometric functions receive due attention. There are sections on component and resultant graphs, on methods of trial and error, on adiabatic curves and harmonic curves, and the decomposition of a curve into harmonic components. The eighth chapter is a very brief discussion of the equations of the conic sections. The simpler and more important properties are embodied in the selection of examples appended to the chapter, so that the student will be familiar with them even should he not be able to study them more intimately. A few useful tables bring to a close a treatise which is comprehensive and thorough, and which by its simplicity of exposition is convincing evidence of Professor Gibson's skill as a teacher.

One hardly knows what to say about *Graphs and Imaginaries*, if the authors intended it for the use of boys. Assuming that it is intended for children of a larger growth, it may be described as an interesting and ingenious piece of work. The sub-title describes its scope—an easy method of finding graphically imaginary roots of quadratic equations and imaginary points of intersections of various curves, with illustrations of the principle from elementary geometry. To solve the ordinary quadratic the parabola $y^2=ax^2+bx+c$ is drawn on squared paper. We are then shown how to draw what the authors call the *shadow* parabola of this curve, which is easily effected, being only the original turned into another position. The unreal roots of the quadratic lie on this *shadow*. The next chapter exhibits the "circle" method of solving a quadratic and the manner in which the shadow may be obtained. From this the imaginary points of intersection of a line and a circle or of two circles readily follow, and the authors then discuss the tangent to a circle from a point within it, and the imaginary points of contact. The relation between the circle and its *shadow* is next treated, and a few problems are worked out, such as:—Divide a line 4 inches long into two parts such that the sum of the squares on them will be 4 square inches; given a circle radius a, find how far distant from the centre is a chord $2c$ units in length, when c is less and when it is greater than a.

Examples in Algebra selected from Elementary Algebra. By W. M. BAKER and A. A. BOURNE. Pp. xi, 222, lxxvi. 3s. Part I., without Answers, 1s. 6d.; Part II., without Answers, 2s. 1904. (Bell.)

The boy or girl who has worked through the whole of the examples given by Messrs. Baker and Bourne should be perfectly familiar with the manipulation of algebraical expressions, and quite at home with the ordinary processes. A word of commendation must be added as to the careful graduation of the

examples. Graphical methods receive due attention, and there is an ample supply of revision papers.

Arithmetical Examples. By W. G. Borchardt. Pp. 247. 3s. 1904. (Rivingtons.)

Exercises in Arithmetic (Oral and Written). Part III. By C. R. Taylor. Pp. viii, 132, xx. 1s. 6d. 1904. (E. Arnold.)

The second edition of Mr. Borchardt's Arithmetical Examples has been enlarged by the addition of many examples of an "easy problem" type, and also of a large number of easy oral questions. They seem to be well graduated, and no doubt will make their way where teachers are beginning to adopt or exhibit a preference for books of examples without an accompanying text-book.

The second book is the third part of the Exercises compiled from the work done by the author in the Wimbledon High School, and follows closely the lines of the Report of the Mathematical Association. They seem carefully selected and graduated. Books such as this and that of Mr. Borchardt represent a considerable amount of patient care and thought, and unless a reviewer has the opportunity of working through much of them in class, he has to be content with expressing a somewhat perfunctory opinion as to their real merits.

Preliminary Practical Mathematics. By S. G. Starling and F. C. Clarke. Pp. 168. 1s. 6d. 1904. (E. Arnold.)

This book is devised for the use of students at technical schools, and seems to be perfectly adapted for the purpose for which it is designed. It contains the irreducible minimum of mathematical knowledge requisite for the intelligent following of the work done in the laboratory and the workshop. It is very depressing to think that such books should be required at all. The same story comes to us from all parts of the country that in these classes, instead of the teacher being able at once to embark on his subject, he has in a large number of cases to teach the would-be student of chemistry or engineering what he ought to have learned at school. As long as this state of affairs persists, we suppose such books as this will be necessary.

Elements of Plane Trigonometry. By R. Lachlan and W. C. Fletcher. Pp. viii, 164. 2s. 1904. (E. Arnold.)

A good deal of skill has been shown in the inclusion within the cover of so small a book of the elements as far as Demoivre's theorem, summation of series, expansions, and factorisations. The readers are assumed to be thoroughly acquainted with the trigonometrical ratios and their applications, so that the introductory pages are to be regarded as an "extension and systematisation of work already done." In the same way, the student is supposed to be acquainted with the use of logarithms. We may draw special attention to the note on Interpolation, p. 12; to the worked out examples on solution of triangles, pp. 76-83, with the hints and observations on the opposite side of the page to the solutions. This book is a bold move in the direction of simplicity, and what the authors have done they have well done.

A School Geometry. Parts I.-VI. **(Containing Plane and Solid Geometry, treated both theoretically and graphically.)** By H. S. Hall and F. H. Stevens. Pp. xiv, 442, ix. 4s. 6d.

A School Geometry. Part VI. **(Containing the substance of Euc. XI., 1-21, together with theorems relating to the surfaces and volumes of the simpler solid figures.)** Pp. 347-442, iv. 1s. 6d. 1904. (Macmillan.)

We have now the whole course of Plane Geometry as set forth by these well-known authors of text-books on elementary mathematics. We notice that enough is shown the student on pp. 270-273 to enable him to proceed at once to the study of such a book as that of Messrs. Lachlan and Fletcher. Let us say that the book is excellent, although it may run on closer lines to Euclid's Elements than is the present fashion in most of the text-books which load our book-shelves. In the eyes of many this will not detract from its merits. The last volume contains figures of the usual nets, with instructions for the making of models therefrom. The "get-up" of the whole series of volumes—paper, printing, and diagrams—is worthy of the firm of Macmillan.

Elementary Geometry of the Straight Line, Circle, and Plane Rectilineal Figures. By CECIL HAWKINS. Part II. Pp. 167-296. 2s. 1904. (Blackie.)

We are afraid that one fatal drawback will be found to the general utility of Mr. Hawkins' admirable edition of the Elements of Geometry. It contains a capital collection of well-devised examples, but no numerical answers are given. With this exception, we can only speak of the book in terms of high praise. We presume that the course he has brought so far forward will be completed by a volume dealing with solid geometry, and in a second edition he might be persuaded to follow the example of others and include a simple treatment of the elementary trigonometrical ratios, etc.

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